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THE EXISTENCE OF BALANCED NEIGHBORLY SPHERES

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Abstract

We concern to the existence of balanced 2-neighborly 3-spheres. The goal of this article is to describe Zheng's results in constructing a balanced 2-neighborly 3-sphere in detail and prove that if Δ is a balanced 2-neighborly 3-sphere then its system of parameters is linear.

Keywords: *Balanced neighborly spheres, balanced complex, linear system of parameters.*

1. Introduction

Let Δ be a simplicial complex on the vertex set $[n] = \{1,2....,n\}$. Thus, Δ is a nonempty collection of subsets of $[n]$ satisfying that $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$. Elements of Δ are called facets of Δ and maximal faces (under inclusion) are called facets. A simplicial complex is called k-neighborly if every subset of vertices of size at most k is the set of vertices of one of its faces. Neighborly complexes, especially neighborly polytopes and spheres are interesting objects to study. In the seminal work of McMullen [1] and Stanley [2], it was shown that in the class of polytopes and simplicial spheres of a fixed dimension and with a fixed number of vertices, the cyclic polytope simultaneously maximizes all the face numbers. The d-dimensional cyclic polytope is $\frac{d}{2}-$ neighborly. Since then, many other classes of neighborly polytopes have been discovered. We refer to [3] and [4] for examples and constructions of neighborly polytopes. Meanwhile, the notion of neighborliness was extended to other classes of objects. For instance, neighborly cubical polytopes were defined and introduced in [5], [6] and neighborly centrally symmetric polytopes and spheres were studied in [7] and [9].

In this paper, we discuss a similar notion for balanced neighborly spheres, which is introduced by H. Zheng in [9]. The author in [9] proved that no balanced 2-neighborly 3-spheres with 12 vertices exist, but with 16 exists. Motivated by this work, the purpose

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of this paper is to describe Zheng's result in constructing a balanced 2-neighborly 3-sphere Γ concretely and give a property about system of parameter of complex Γ. The structure of this article is organized as follows. In Section 2, we discuss basic propertiesof balanced neighborly spheres and give some examples to illustrate it. In Section 3, we present how to construct balanced 2-neighborly 3-sphere and describe it in detail. And prove that if Γ is a balanced 2-neighborly 3-sphere then system Θ is a linear system of parameters of K_Δ in Proposition 3.2.

2. Balanced neighborly spheres

In this section we introduce balanced neighborly sphere and some notations that will be used throughout this article. We first recall basic definitions on spheres. The dimension of a face $F \in \Delta$ is dim F = |F| - 1 and the dimension of Δ is the maximum dimension of its faces. Faces of dimension 0 are called vertices and faces of dimension 1 are called edges. A map k:$[n] \to [d]$ is called a proper coloring map of Δ, if we have $\kappa(i) \neq \kappa(i)$ for any edge $\{i, j\} \in \Delta$. We say that a (d - 1) - dimensional simplicial complex Δ on $[n]$ is balanced (completely balanced in some literature) if its graph is d-colorable.

Example 2.1. We have

dim Δ_1 = 1, dim Δ_2 = 2

Since Δ_1 is 2-colorable and Δ_1 is 3-colorable. So Δ_1, Δ_2 are balanced complex.

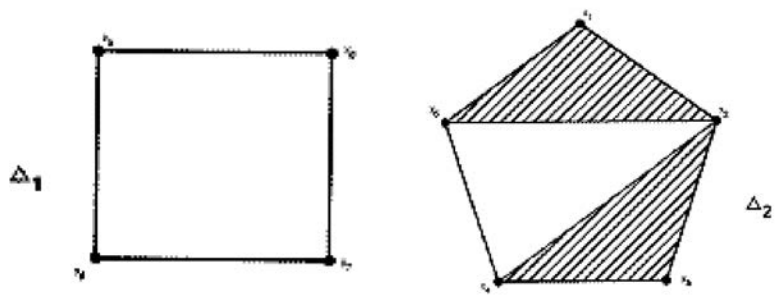

Figure 1. Complex C4 and C5

A simplicial complex is pure if all of its facets (maximal faces) have the same dimension. The geometric realization of Δ, $|\Delta|$ is the union in $\mathbb{R}^n$ over all faces $\{u_{i1},...,u_{ij}\}$ of Δ of the convex hull of $\{e_{i1},...,e_{ij}\}$, where $\{e_1,...,e_n\}$ is the standard basis of $\mathbb{R}^n$. We say that Δ is it homeomorphic to another space whenever $|\Delta|$ is. A triangulation of a topological space T is any simplicial complex Δ such that $|\Delta|$ is homeomorphic to T. For a face $F \in \Delta$, the subcomplexes

$$lk_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \varnothing\}$$

and

$$st_\Delta(F) = \{G \subset \Delta : F \cup G \in \Delta\}$$

is called the link and star of F in Δ. If Δ is a pure (d - 1) - dimensional complex such that every (d - 2) - dimensional face of Δ is contained in at most two facets, then the boundary complex of Δ, $\partial\Delta$, consists of all (d - 2)-dimensional faces that are contained in exactly one facet, as well as their subsets. A simplicial complex Δ is a simplicial sphere (resp. simplicial ball) if the geometric realization of Δ is homeomorphic to a sphere (resp. ball). The boundary complex of a simplicial d-ball is a simplicial (d - 1)-sphere. A balanced simplicial complex is called balanced k-neighborly if every set of k or fewer vertices with distinct colors forms a face. The following Lemma plays an important role in find balanced neighborly sphere in next section.

Lemma 2.2. [9] Let d > 4. If Δ is balanced homology (d - 1)-sphere and $V_d = \{v_1, v_2, v_3\}$ is the set of vertices of color d, then $lk_\Delta v_i \bigcap lk_\Delta v_j$ is a homology (d - 2)-ball for any $1 \le i < j \le 3$, and $\bigcap_{i=1}^{k} lk_\Delta v_i$ is a homology (d - 3)-sphere.

3. The existence of balanced 2-neighborly 3-sphere

In this section, we present how to construct balanced 2-neighborly 3- sphere and describe it in detail. Assume that V_1 = {u_1; u_2; u_3; u_4;}, V_2 = {v_1; v_2; v_3; v_4;}, V_3 = w_1; w_2; w_3; w_4;} and V_4 = {z_1; z_2; z_3; z_4;} are the four color sets of a balanced 3-sphere Γ. Let $lk_\Gamma z_1 = A \bigcup_{\partial A \sim \partial C} C$ and $lk_\Gamma z_3 = B \bigcup_{\partial A \sim \partial C} C$, where A, B and C are triangulated 2-balls sharing the same boundary as shown in Figure 2.

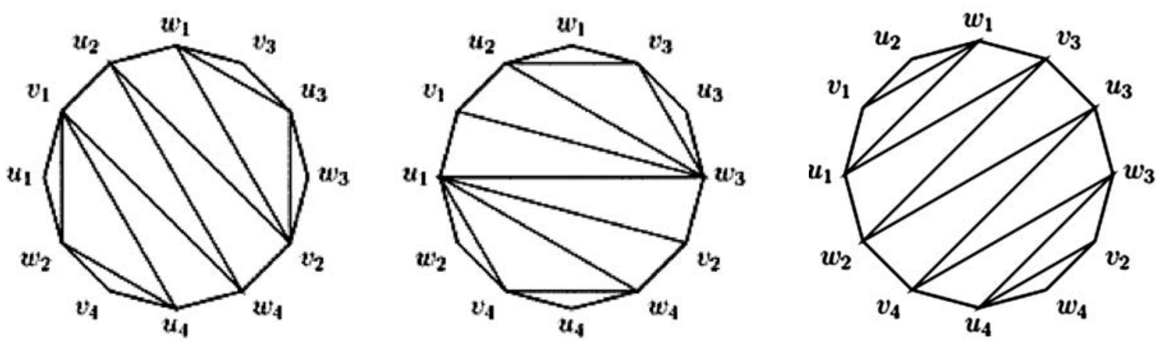

Figure 2. Discs A, B and C (from left to right)

All possible edges that do not appear in A, B and C are shown in Figure 3 as solid red edges in disc D'. Notice that the dashed edges in D' are edges in discs A and B, so we may rearrange the boundary of D by switching the positions of vertices v_1 and v_2 and then replacing the edges containing v_1 or v_2 in $\partial D'$ by the dashed edges. In this way, we obtain a triangulation of a 12-gon D as shown in Figure 3.

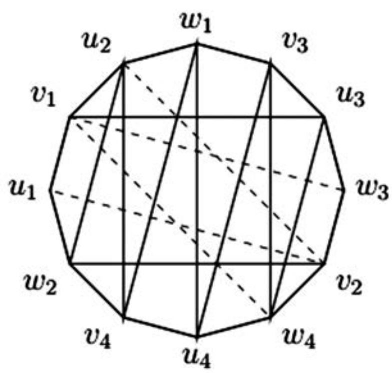

Figure 3. Left: disc *D'*. Right: disc D obtained after rearranging the boundary of *D'*

Furthermore, $\partial D \subseteq A\bigcup B$ and ∂D divides the sphere $A \bigcup\limits_{\partial A\sim\partial B} B$ into two discs A' and B' as shown in Figure 4. Let $lk_\Gamma z_2 = A' \bigcup\limits_{\partial A'\sim\partial D} D$ and $lk_\Gamma z_4 = B' \bigcup\limits_{\partial B'\sim\partial D} C$. Since both $st_\Gamma z_1 \bigcup st_\Gamma z_3 = C$ and $st_\Gamma z_2 \bigcup \left(st_\Gamma z_1 \bigcup st_\Gamma z_3\right) = A'$ are simplicial 2-balls, it follows that $M = \bigcup\limits_{i=1}^{3} st_\Gamma z_i$ is a simplicial 3-ball. Furthermore, the boundary of M is exactly $lk_\Gamma z4$. Hence $\Gamma = M \bigcup st_\Gamma z_4$ is indeed a balanced 2-neighborly 3-sphere.

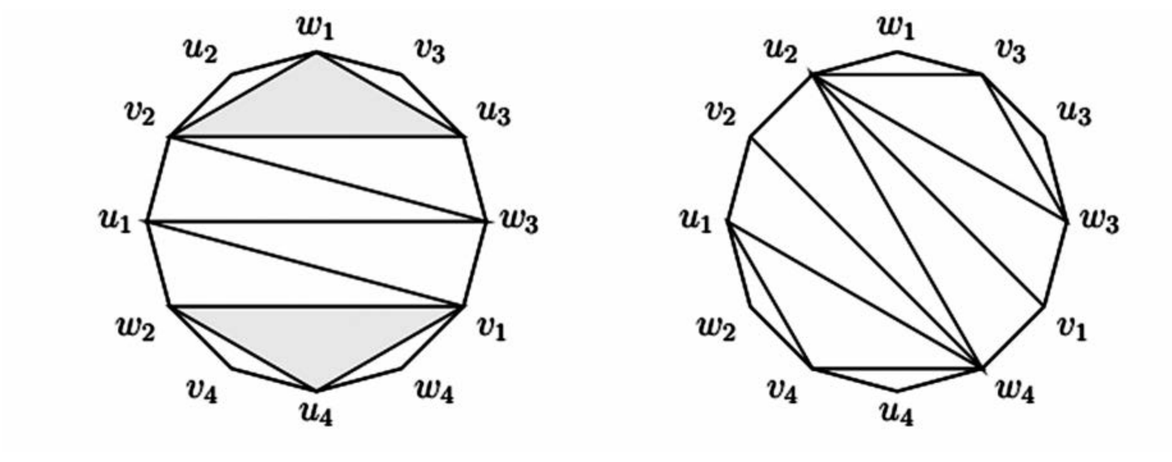

Figure 4. Left: disc *A'*. Right: disc *B'*. Notice that ∂A' = ∂B' = ∂D

Thus, we can describe balanced 2-neighborly 3-sphere as follows:

$$
\begin{aligned}
\Gamma = \{ & z_1u_1v_1w_2; z_1u_4v_1w_2; z_1u_4v_4w_2; z_1u_4v_1w_4; z_1u_2v_1w_4; z_1u_2v_2w_4; z_1u_2v_2w_1; z_1u_3v_2w_1; \\
& z_1u_3v_2w_3; z_1u_3v_3w_1; z_1u_2v_1w_1; z_1u_1v_1w_1; z_1u_1v_3w_1; z_1u_1v_3w_2; z_1u_3v_3w_2; z_1u_3v_4w_2; \\
& z_1u_3v_4w_3; z_1u_4v_4w_3; {}_1u_4v_2w_3; z_1u_4v_2w_4; z_3u_2v_3w_1; z_3u_2v_3w_3; z_3u_3v_3w_3; z_3u_2v_1w_3; \\
& z_3u_1v_1w_3; z_3u_1v_2w_3; z_3u_1v_2w_4; z_3u_1v_4w_4; z_3u_1v_4w_2; z_3u_4v_4w_4; z_3u_2v_1w_1; z_3u_1v_1w_1; \\
& z_3u_1v_3w_1; z_3u_1v_3w_2; z_3u_3v_3w_2; z_3u_3v_4w_2; z_3u_3v_4w_3; z_3u_4v_4w_3; z_3u_4v_2w_3; z_3u_4v_2w_4; \\
& z_2u_2v_2w_1; z_2u_3v_3w_1; z_2u_3v_2w_1; z_2u_3v_2w_3; z_2u_1v_2w_3; z_2u_1v_1w_3; z_2u_1v_1w_2; z_2u_4v_1w_2; \\
& z_2u_4v_4w_2; z_2u_4v_1w_4; z_2u_1v_2w_2; z_2u_2v_2w_2; z_2u_2v_4w_2; z_2u_2v_4w_1; z_2u_4v_4w_1; z_2u_4v_3w_1; \\
& z_2u_4v_3w_4; z_2u_3v_3w_4; z_2u_3v_1w_4; z_2u_3v_1w_3; z_4u_2v_3w_1; z_4u_2v_3w_3; z_4u_3v_3w_3; z_4u_2v_1w_3; \\
& z_4u_2v_1w_4; z_4u_2v_2w_4; z_4u_1v_2w_4; z_4u_1v_4w_4; z_4u_1v_4w_2; z_4u_4v_4w_4; z_4u_1v_2w_2; z_4u_2v_2w_2; \\
& z_4u_2v_4w_2; z_4u_2v_4w_1; z_4u_4v_4w_1; z_4u_4v_3w_1; z_4u_4v_3w_4; z_4u_3v_3w_4; z_4u_3v_1w_4; z_4u_3v_1w_3 \}
\end{aligned}
$$

Let Δ be a (d - 1)-dimensional balanced simplicial complex on [n] and let $R = K[x_1,..., x_n$be a polynomial ring over an infinite field K. The ring $K[\Delta] = R/I_\Delta$, where $I_\Delta = \left(\prod_{i\in F} x_i \mid F \subset [n], F \notin \Delta\right)$, is called the Stanley-Reisner ring of Δ.

Example 3.1. Consider complex Δ_1 and Δ_2 as Figure 1. We have

$$I_{\Delta_1} = (x_1, x_3, x_2, x_4),\ \dim K[\Delta_1] = 2$$

$$I_{\Delta_2} = (x_1, x_3, x_1x_4, x_3x_5, x_2x_3x_4),\ \dim K[\Delta_2]=3$$

Set $Vk = \{v \in [n] | k(v) = k\}$. Let $\theta_k = \sum_{v\in V_k} x_{v_k}$, for $k = 1,2,...,d$. Then $\Theta = \theta_1,...,\theta_d$ is a system of parameters of $K[\Delta] = R/I_\Delta$. The Krull dimension of $K[\Delta]$ is the minimal number k such that there is a sequence $\theta_1,...,\theta_k \in R$ of linear forms such that $\dim_k K[\Delta]/(\Theta)<\infty$. It is well-known that the Krull dimension of $K[\Delta]$ equals $\dim \Delta +1$ ([10], II Theorem 1.3). If $K[\Delta]$ is of Krull dimension d, then a sequence Θ of linear forms such that $\dim_k K[\Delta]/(\Theta)<\infty$ is called a linear system of parameters (l.s.o.p. for short) of $K[\Delta]$. Therefore, we have the following property.

Proposition 3.2. If Γ is a balanced 2-neighborly 3-sphere then Θ is a linear system of parameters of $K[\Delta]$.

Proof. We see that $\dim_k K[\Gamma] = 4$ and

$$\Theta = \{z_1 + z_2 + z_3 + z_4, u_1 + u_2 + u_3 + u_4, v_1 + v_2 + v_3 + v_4, w_1 + w_2 + w_3 + w_4\}.$$

Therefore, $\dim_k K[\Delta]/(\Theta) = 3$ and then Θ is linear system of parameters of $K[\Delta]$.

4. Conclusions

The main result of the article is to describe Zheng's result in constructing a balanced 2-neighborly 3-sphere in detail and prove that if Δ is a balanced 2-neighborly 3-sphere then its system of parameters is linear.

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SỰ TỒN TẠI CỦA CẦU CÂN BẰNG NEIGHBORLY

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Tóm tắt

Chúng tôi quan tâm đến sự tồn tại của cầu đơn hình cân bằng neighborly. Mục tiêu của bài báo là mô tả một kết quả của H. Zheng về việc xây dựng một 3-cầu cân bằng 2-neighborly một cách chi tiết và chứng minh rằng nếu Γ là 3 cầu cân bằng 2-neighborly thì hệ tham số của nó là tuyến tính.

Từ khóa: *Cầu đơn hình cân bằng neighborly, phức cân bằng, hệ tham số tuyến tính.*